During the computation of the results in Fig. 1, each velocity $\lambda$ was first included in the aerodynamic matrix which was subsequently combined with the stiffness matrix, the Euler buckling load was then calculated with the neglect of the mass matrix, the postbuckling behavior was finally predicted by the piecewise linear incremental procedure.

In Fig. 1, it is seen that in the region where $\Delta T/\Delta T_{cr}<1$, no buckling deflection will occur. In the region where $1>\Delta T/\Delta T_{cr}<2.485$, an increase in the velocity $\lambda$ will increase the pressure of the airstream which tends to stabilize or blow flat the buckled panel. In the region where $\Delta T/\Delta T_{cr}>2.485$, an increase in velocity $\lambda$ will reduce the buckled depth until it reaches the value of $1.180\pi^4$. When the velocity reaches beyond this value, the static instability will occur. Mathematically speaking, when $\lambda>1.180\pi^4$ the eigenvalue for the Euler buckling temperature rise $\Delta T$ becomes imaginary whereas a real solution is not obtainable.

The stability boundaries for the simply-supported buckled panels can be presented in a simpler form by using the data obtained in Fig. 1 with the neglect of the deflection parameter. The boundary is shown as the lower curve in Fig. 2. The case of clamped edge condition is also considered and the results are also plotted in Fig. 2. Both curves are slightly higher than the ones obtained by the Galerkin's two-mode approximation. It is seen in Fig. 2 that as long as the velocity stays below the horizontal line for each panel, the buckled panel will always be stable.

An observation of the results in Fig. 2 reveals that in order for dynamic instability to occur for $\Delta T/\Delta T_{cr}$ less than 2.485 and 6.161 for the simply-supported and clamped panels, respectively, the panels must be blown flat or unbuckled. The dynamic instability boundaries for various values of temperature rise $\Delta T$ were obtained and shown as the two upper curves in Fig. 2. Such curves were found by first specifying the value of $\Delta T$ and then varying the value of flow velocity $\lambda$ in small increment until the eigenvalue for first mode frequency changes from real to complex. It is seen that these results are in close agreement with the exact solutions.[7-9]

## Conclusions

The basic procedure for the flutter analysis of flat finite element panels with elastic boundary constraints and subjected to temperature rise is outlined. The aerodynamic theory is based on the piston theory with first-order approximation. The Mach number considered is limited to be beyond approximately 1.6. With the use of a linear incremental procedure combined with a coordinate transformation technique, the postbuckling behavior of a heated panel under the stabilizing effect of airstream pressure can be predicted and the static instability boundary can be found. The dynamic instability boundary for the flat panel can also be found.

This basic procedure can easily be extended to the general panel system for practical application in the aeronautical engineering industry. The general cases can be rectangular panels of finite aspect ratio; panels of delta or other arbitrary shapes; panels with cut-outs; complex elastic edge conditions; stiffened and composite panels; and slightly curved panels, etc. All these panels can be subjected to the aerodynamic heating.

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# Theory of Adjoint Structures


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## Nomenclature

$|A|'$ = transpose of $|A|$
$|B|'$ = complex conjugate of $|B|$
$|C|^*$ = complex tranjugate of $|C|$

## I. Introduction

MECHANICAL structures usually are designed in an iterative way. The design procedure starts with an "as good as possible" intuitive design of the structure. Then, the structure is analyzed to check if the design specifications are met. If this is not so, the design must be improved and analyzed again. This cycle is repeated until the specifications are met in the best way. The automatic design of a mechanical structure can be reduced to the minimizing of a performance function. The performance function is a measure of the deviation between the actual and desired behavior of the structure. To decrease the number of iteration steps it is possible to use an optimization strategy. In many important optimization methods, the most effective improvements are derived from the gradients of the performance function.[1] These gradients or sensitivities give the influence of the building element parameters on the performance function of the whole mechanical structure.

In the classical methods of sensitivity analysis it is assumed that the structures are linear and statically or kinematically determinate.[2-6] This supposition makes the calculation of the sensitivities very easy. The stiffness matrix of a linear structure, for instance, is equal to

$$|K'| = |B|'|K|\,|B| \tag{1}$$

where

$|K|$ = stiffness matrix of the building elements, considered together but not connected
$|B|$ = compatibility conditions

If the structure is kinematically determinate, the sensitivity of the stiffness matrix for the change of a parameter $R_k$ of a building element is given by

$$\partial|K'|/\partial R_k = |B|'(\partial|K|/\partial R_k)\,|B| \tag{2}$$

Indeed, the compatibility conditions and the matrix $|B|$ are not influenced by the parameters $R_k$. This is not the case for


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kinematically indeterminate structures, so that the sensitivity analysis becomes more complex. Director and Rohrer[7] developed a general method for the sensitivity analysis of electrical networks. This method, called the adjoint network theory, is based on Tellegen's theorem.[8] The analogy between Tellegen's theorem and the theorem of virtual work makes it possible to extend the adjoint network theory to mechanical structures.[9]

## II. Generalized Theorem of Virtual Work

Consider the structure of Fig. 1. This structure consists of $m$ mutually connected building elements. The connections and building elements are taken in separate black boxes on the $n$--port representation. If $|P'|$ indicates the vector of forces and moments, and $|Q'|$ the vector of displacements and rotations acting or appearing on the complete structure; and $|P_i|$ represents the vector of forces and moments, and $|Q_i|$ the vector of displacements and rotations acting or appearing on the $i$th building elements, then we can write, for small displacements and rotations, that

$$|P'|^t |Q'| = \sum_{i=1}^{m} |P_i|^t |Q_i| \tag{3}$$

This equation expresses the classical form of the theorem of virtual work.

The theorem of virtual work can be generalized further. For that purpose we consider, besides the given structure, a second structure, the adjoint structure of Fig. 2. The building elements of the structure just given and the adjoint structure are connected in the same manner, and the nodal points have corresponding geometric positions. The parameters of the building elements may be different. If the displacements and rotations are sufficiently small, it is possible to show that

$$|\hat{P}'|^t |Q'| = \sum_{i=1}^{m} |\hat{P}_i|^t |Q_i| \tag{4}$$

The proof is based only on the equilibrium and compatibility conditions. A linear behavior of the building elements is not required. The generalized theorem of virtual work constitutes the starting point of the theory of adjoint structures.

## III. Theory of Adjoint Structures

Suppose that the building element parameters of the structure represented in Fig. 3 must be determined.

The input variables are $|P'_x|$ = the applied forces and moments and $|Q'_x|$ = the imposed displacements and rotations. As output variables, we have $|Q'_y|$ = the displacements and rotations at the place of the applied forces and moments, and $|P'_y|$ = the forces and moments at the place of the imposed displacements and rotations.

The performance function is a scalar function which is larger than or equal to zero, and reaches an absolute minimum if the design specifications are met. If $|\bar{Q}'_y|$ indicates the desired displacements and rotations, and $|\bar{P}'_y|$ the desired forces and moments, and if $|W_q|$ represents the diagonal matrix of the weighting factors of displacements and rotations, and $|W_p|$ the diagonal matrix of the weighting factors of forces and moments, then the performance function $\xi$ can be defined by

$$\xi = \tfrac{1}{2}\left[ |Q'_y| - |\bar{Q}'_y| \right]^* |W_q| \left[ |Q'_y| - |\bar{Q}'_y| \right] + \tfrac{1}{2}\left[ |P'_y| - |\bar{P}'_y| \right]^* |W_p| \left[ |P'_y| - |\bar{P}'_y| \right] \tag{5}$$

To calculate the sensitivities in a simple way, an additional structure, the adjoint structure of Fig. 4, is introduced. The adjoint structure has the topology and geometry of the given structure. The building elements are, however, linear and

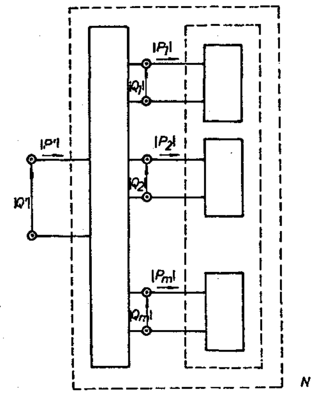

**Fig. 1 $n$-port representation of a structure consisting of $m$ mutually connected building elements.**

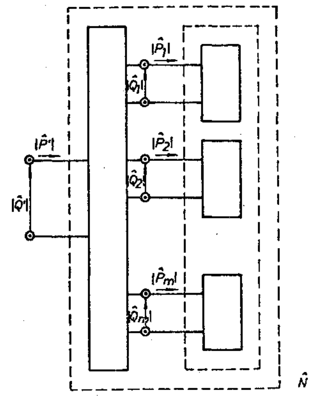

**Fig. 2 Adjoint structure. Structure with the topology and geometry of the structure of Fig. 1. The building element parameters may be different.**

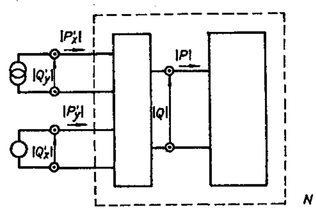

**Fig. 3 Structure in which building element parameters have to be optimized.**

characterized by the following:

Flexibility Matrix

$$|\hat{S}| = \partial |Q| / \partial |P| \tag{6}$$

Stiffness Matrix

$$|\hat{K}| = \partial |P| / \partial |Q| \tag{7}$$

The input variables of the adjoint structure are equal to

$$|\hat{P}'_x| = |W_q| \left[ |Q'_y| - |\bar{Q}'_y| \right]' \tag{8}$$

$$|\hat{Q}'_x| = -|W_p| \left[ |P'_y| - |\bar{P}'_y| \right]' \tag{9}$$

It now is possible to prove that the sensitivity of the performance function for a change of a real parameter $R_k$ is expressed by

$$\frac{\partial \xi}{\partial R_k} = Re\left[ |\hat{P}|^t \frac{\partial |Q|}{\partial R_k} \right] \tag{10}$$

or

$$\frac{\partial \xi}{\partial R_k} = -Re\left[ |\hat{Q}|^t \frac{\partial |P|}{\partial R_k} \right] \tag{11}$$

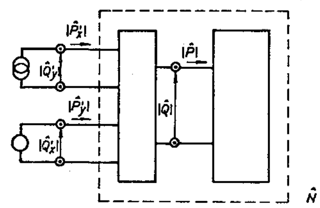

**Fig. 4 Adjoint structure introduced to simplify the sensitivity analysis of the structure of Fig. 3.**

The sensitivities $\partial |Q|/\partial R_k$ and $\partial |P|/\partial R_k$ can be derived from the mathematical models of the building elements of the given structure. Linearity of the building elements is not necessary.

If the building elements are linear, Eqs. (6) and (7) may be reduced to

$$|\hat{S}| = |S|' \tag{12}$$

$$|\hat{K}| = |K|' \tag{13}$$

Equations (10) and (11) are simplified to

$$\frac{\partial \xi}{\partial R_k} = Re\left[ |\hat{P}|' \frac{\partial |S|}{\partial R_k} |P| \right] \tag{14}$$

or

$$\frac{\partial \xi}{\partial R_k} = -Re\left[ |\hat{Q}|' \frac{\partial |K|}{\partial R_k} |Q| \right] \tag{15}$$

These equations still are valuable if the given structure is statically or kinematically indeterminate.

Similar simple formulas can be derived for the calculation of sensitivities of stiffness and flexibility matrices of linear structures. They can be applied for the sensitivity analysis of natural frequencies and mode shapes. In these cases, only one analysis is sufficient to calculate all sensitivities.[9]

## IV. Conclusions

The theory of adjoint structures gives us a general, but still simple, method for the sensitivity analysis of mechanical structures. After one analysis of the given and one analysis of the adjoint structure, it is possible to calculate all sensitivities easily. As shown in this paper, the analogy between electrical networks and mechanical structures allows a fruitful exchange of methods and techniques between both fields of computer-aided design. Even a common and general approach can be developed.[9]

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# Transient Temperature Distortion in a Slab Due to Thermocouple Cavity


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## Introduction

IN heat transfer studies, many experimental difficulties may arise if heat flux sensors or thermocouples are installed directly on the surface of a body. A piston sliding in a cylinder, a projectile moving in a barrel, and the melting or ablation of a heat shield are some examples. In these cases, the transient surface temperature and heat flux may be determined by inverting the temperature measured by a probe located under the surface of the solid material. This inversion problem has been studied theoretically by Chen and Thomsen,[1] Imber,[2] Sparrow et al.,[3] Stolz,[4] Beck,[5] and Frank[6] for various geometry and boundary conditions. Most of these analyses assumed a one-dimensional model, but in reality the temperature field is distorted to become two or three dimensional when a cavity is drilled to accommodate the thermocouple leads. The degree of distortion may be influenced by the dissimilar properties of the thermocouple and the surrounding material, and by the diameter and depth of the cavity.

Chen and Thomsen[1] showed that the error, particular in the transient case, will be amplified when the measured data are inverted for a prediction of surface heat flux. Therefore, in the present study, an experiment was performed in the transient period of heat conduction for a slab subjected to a constant heat flux at one surface and insulated on the other surface. The deviation of temperature response resulting from the presence of a cavity is given as a function of the diameter and depth of the cavity, so that a proper correction to the one-dimensional model can be made. The Fourier number, or dimensionless time $\tau = \alpha t/L^2$, where $t$ is the dimensional time, varied from 0 to 0.5. Alumel-chromel thermocouples with a diameter of $d' = 0.0203$ cm were embedded in the slab. They were used throughout the experiment to determine how a geometric disturbance causes the measured transient temperature to deviate from that of an undisturbed, one-dimensional response.

## Experiment

The experiment is set up as shown in Fig. 1. An electrical oven heated to 852°C and having a 24×19 cm opening was used as the heating source. The door was designed so that it could slide quickly, within 0.3 sec., up the test piece into the test position. Three test pieces with thicknesses of $L = 1.27$, 1.91, and 2.54 cm were made of hot-rolled steel (AISISAE 1020) with thermal conductivity, $K = 0.519$ w/cm°C, and thermal diffusivity of $\alpha = 0.08$ cm$^2$/sec.[7] Each test piece was coated with carbon black on the surface facing the oven and framed with a strip of asbestos around the edge and outer surfaces to prevent heat loss during the experiment. Each test piece also had four cavities within the dimensionless distances from the heated surface, $S/L$, being 0.2, 0.4, 0.6, and 0.8. The drill point half-angle of the drill bit used to form the cavities was 54°. At least 10 cm distance was kept between cavities. For this distance, the lateral distortions of temperature caused by the cavities did not appear to interfere with


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